NOTE ON RADIAL VIBRATIONS OF NON-HOMOGENEOUS SPHERICAL AND CYLINDRICAL SHELLS

S. DE

DEPARTMENT OF PHYSICS, VISVA-BHARATI, INDIA.

(*Received May* 23, 1968)

ABSTRACT. In this paper the radial vibrations of spherical and cylindrical shells characterised by a particular type of non-homogeneity have been discussed.

INTRODUCTION

The first case deals with the radial vibration of a non-homogeneous spherical shell whose internal radius is a and external radius is b. In the second case we have considered the radial vibration of anon-homogeneous cylindrical shell with c and d as inner and outer radii of the shell respectively. In both the cases we have taken some power law variation of elastic constants and have supposed that the density is constant. It is believed that the particular case has not been discussed by any previous investigator.

PROBLEM AND ITS SOLUTION

Case—I : We suppose that the displacements u_θ, u_φ vanish and $u_r(=u)$ is a function of r only. The stress-components are given by (Love, *p*. 142, 1944)

$$\left.\begin{aligned} \widehat{rr} &= (\lambda+2\mu)\frac{\partial u}{\partial r}+2\lambda\frac{u}{r} \\ \widehat{\theta\theta} &= \widehat{\phi\phi} = \lambda\frac{\partial u}{\partial r}+2(\lambda+\mu)\frac{u}{r} \\ \widehat{\theta\phi} &= \widehat{\phi r} = \widehat{r\theta} = 0 \end{aligned}\right\} \quad \dots \quad (1)$$

The stress equations of motion are (Love, 1944)

$$\left.\begin{aligned} &\frac{\partial}{\partial r}\widehat{rr}+\frac{1}{r}\frac{\partial}{\partial\theta}\widehat{r\theta}+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\widehat{r\phi}+\frac{1}{r}(2\widehat{rr}-\widehat{\theta\theta}-\widehat{\phi\phi}+\widehat{r\theta}\cot\theta) = \rho\frac{\partial^2 u_r}{\partial t^2}, \\ &\frac{\partial}{\partial r}\widehat{r\theta}+\frac{1}{r}\frac{\partial}{\partial\theta}\widehat{\theta\theta}+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\widehat{\theta\phi}+\frac{1}{r}\{(\widehat{\theta\theta}-\widehat{\phi\phi})\cot\theta+3\widehat{r\theta}\} = \rho\frac{\partial^2 u_\theta}{\partial t^2}, \\ &\frac{\partial}{\partial r}\widehat{r\phi}+\frac{1}{r}\frac{\partial}{\partial\theta}\widehat{\theta\phi}+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\widehat{\phi\phi}+\frac{1}{r}\{3\widehat{r\phi}+2\widehat{\theta\phi}\cot\theta\} = \rho\frac{\partial^2 u_\varphi}{\partial t^2}. \end{aligned}\right\} \quad \dots \quad (2)$$

K

Assuming $\lambda = \lambda_0 r^2$, $\mu = \mu_0 r^2$ and $\rho = \rho_0$, λ_0, μ_0, ρ_0 being constants and substituting equations (1) in equations (2) we find that the second and third equations are identically satisfied. From the first equation of (2), we get

$$(\lambda_0+2\mu_0)r^2 \frac{\partial^2 u}{\partial r^2} + r[2(\lambda_0+2\mu_0)+2(\lambda_0+2\mu_0)]\frac{\partial u}{\partial r}+(2\lambda_0-4\mu_0)u = \rho_0 \frac{\partial^2 u}{\partial t^2}.$$

Assuming $\lambda_0 = \mu_0$ (Poisson's condition) and taking $u = Ue^{i(pt+\epsilon)}$, the above equation reduces to

$$r^2 \frac{\partial^2 U}{\partial r^2} + 4r \frac{\partial U}{\partial r} - \frac{2}{3} U + \frac{\rho_0 p^2}{3\mu_0} U = 0. \quad \text{Putting} \quad k^2 = \frac{\rho_0 p^2}{3\mu_0}$$

whe have
$$\frac{\partial^2 U}{\partial r^2} + \frac{4}{r} \frac{\partial U}{\partial r} + \left(\frac{3k^2-2}{3r^2}\right) U = 0 \qquad \ldots (3)$$

The solution of the above equation is given by

$$U = Ar^{m_1}+Br^{m_2},$$

where
$$m_1 = (-9+\sqrt{105-36k^2})/6$$

and
$$m_2 = (-9-\sqrt{105-36k^2})/6;$$

A and B being any two arbitrary constants.

Therefore, $u = (Ar^{m_1}+Br^{m_2})e^{i(pt+\epsilon)}$

Now,
$$\widehat{rr} = 3\mu_0 r^2 \frac{\partial u}{\partial r} + 2\mu_0 r u$$

$$= \mu_0 e^{i(pt+\epsilon)}[(3m_1+2)Ar^{m_1+1}+(3m_2+2)Br^{m_2+1}].$$

Boundary conditions : We assume that $\widehat{rr} = 0$

at $r = a$ and $r = b$.

Therefore,
$$(3m_1+2)Aa^{m_1+1}+(3m_2+2)Ba^{m_2+1} = 0 \qquad . (4)$$

and
$$(3m_1+2)Ab^{m_1+1}+(3m_2+2)Bb^{m_2+1} = 0 \qquad . (5)$$

Eliminating A and B from equations (4) and (5) we get

$$(3m_1+2)a^{m_1+1}(3m_2+2)b^{m_2+1}-(3m_1+2)b^{m_1+1}(3m_2+2)a^{m_2+1} = 0$$

or,
$$a^{m_1+1}.\, b^{m_2+1}-b^{m_1+1}.\, a^{m_2+1} = 0. \qquad (6)$$

This is the frequency equation for the radial vibration of the spherical shell. From equation (6), we get

$$m_1 = m_2$$

or,
$$105-36k^2 = -(105-36k^2).$$

Therefore,
$$k^2 = \frac{105}{36}$$

Hence,
$$\frac{105}{36} = \frac{\rho_0 p^2}{3\mu_0}$$

Therefore,
$$p = \sqrt{8.75\frac{\mu_0}{\rho_0}}.$$

Case—II. We take the axis of the cylindrical shell as the axis of z and we assume that $v = 0$, $w = 0$ and u is independent of θ and z, where u, v, w are the components of displacement in cylindrical coordinates.

The stress components in cylindrical coordinates are

$$\left.\begin{aligned}
\widehat{rr} &= (\lambda+2\mu)\frac{\partial u}{\partial r}+2\lambda\frac{u}{r}\\
\widehat{\theta\theta} = \widehat{zz} &= \lambda\frac{\partial u}{\partial r}+2(\lambda+\mu)\frac{u}{r}\\
\widehat{\theta z} = \widehat{zr} &= \widehat{r\theta} = 0
\end{aligned}\right\} \tag{7}$$

The stress equations of motion in terms of displacements are (Love 1944, p. 90)

$$\left.\begin{aligned}
&\frac{\partial}{\partial r}\widehat{rr}+\frac{1}{r}\frac{\partial}{\partial\theta}\widehat{r\theta}+\frac{1}{r}(\widehat{rr}-\widehat{\theta\theta}) = \rho\ddot{u}\\
&\frac{\partial}{\partial r}\widehat{r\theta}+\frac{1}{r}\frac{\partial}{\partial\theta}\widehat{\theta\theta}+\frac{\partial}{\partial z}\widehat{\theta z}+\frac{2}{r}\widehat{r\theta} = \rho\ddot{v}\\
&\frac{\partial}{\partial r}\widehat{rz}+\frac{1}{r}\frac{\partial}{\partial\theta}\widehat{\theta z}+\frac{\partial}{\partial z}\widehat{zz}+\frac{1}{r}\widehat{rz} = \rho\ddot{\omega}
\end{aligned}\right. \qquad \ldots \tag{8}$$

We take
$$\lambda = \lambda_0 r^2, \quad \mu = \mu_0 r^2$$

and
$$\rho = \rho_0, \quad \lambda_0, \mu_0, \rho_0 \text{ being constants.}$$

Substituting equations (7) in (8) we find that the second and third equations of (8) are identically satisfied. The first equation gives

$$r^2 \frac{\partial^2 u}{\partial r^2}(\lambda_0+2\mu_0)+r[2(\lambda_0+2\mu_0)+2(\lambda_0+\mu_0)]\frac{\partial u}{\partial r}+(2\lambda_0-2\mu_0)u = \rho_0 \frac{\partial^2 u}{\partial t^2}$$

Taking $\lambda_0 = \mu_0$, (Poisson's condition) and assuming $u = Ue^{i(pt+\epsilon)}$, we have

$$r^2 \frac{\partial^2 U}{\partial r^2} \cdot 3\mu_0 + r \frac{\partial U}{\partial r} 10\,\mu_0 + \rho_0 p^2 U = 0$$

or,
$$\frac{\partial^2 U}{\partial r^2} + \frac{10}{3r}\frac{\partial U}{\partial r} + \frac{k^2}{r^2} U = 0, \tag{9}$$

where
$$k^2 = \frac{\rho_0 p^2}{3\mu_0}$$

The solution of equation (9) is given by

$$U = Ar^{m_3}+Br^{m_4}, \quad \text{where}$$

$$m_3 = (-7+\sqrt{49-36k^2})/6$$

and
$$m_4 = (-7-\sqrt{49-36k^2})/6;$$

A and B are any two arbitrary constants.

Therefore,
$$u = (Ar^{m_3}+Br^{m_4})e^{i(pt+\epsilon)}$$

We assume that $\widehat{rr} = 0$ at $r = c$ and $r = d$.

Proceeding like case-I, the frequency equation for the radial vibration of the cylindrical shell can be written as

$$c^{m_3+1} \cdot d^{m_4+1} - d^{m_3+1} \cdot c^{m_4+1} = 0, \tag{10}$$

From equation (10), we get

$$m_3 = m_4 \quad \text{or,} \quad 49-36k^2 = -(49-36k^2)$$

Therefore,
$$k^2 = \frac{49}{36}$$

Hence,
$$\frac{49}{36} = \frac{\rho_0 p^2}{3\mu_0}$$

Therefore,
$$p = 3.5\sqrt{\frac{\mu_0}{3\rho_0}}$$

REFERENCE

Love, A. E. H., 1944, *A Treatise on the Mathematical theory of Elasticity*, Dover Publications.